ON THE INSTABILITY OF THE MOTION OF SYSTEMS WITH RETARDATION

(O NEUSTOICHIVOSTI DVIZHENIIA SISTEM S ZAPAZDYVANIEM PO VREMENI)

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In this paper it is shown that the known theorems of Liapunov [1] and of Chetaev [2] concerning stability may be extended to systems with retardation. A criterion of instability in first approximation of motion of systems with retardation is given.

1. Chetaev's theorem on the instability of motion, and its application to systems with retardation. Consider equations of perturbed motion of the form

$$\frac{dx_i(t)}{dt} = X_i(x_1(t+\vartheta),\ldots,x_n(t+\vartheta),t) \qquad (i=1,\ldots,n) \tag{1.1}$$

where the $X_i(x_1(\theta), \ldots, x_n(\theta), t)$ are functionals defined for any piece-wise-continuous (i.e., having, at worst, discontinuities of the first kind) functions $x_i(\theta)$ defined on the interval $-r \leqslant \theta \leqslant 0$, and one has $X_i(0, \ldots, 0, t) \equiv 0$.

Equations (1.1) are the general form of the equations with retardation, and are called equations with after-effects.

In order to determine the derivatives $dx_i(t)/dt$ at a given t, it is necessary to know not only the $x_i(t)$ at the instant t, but to know them at all instants t on the interval $[t-r, t]$. Consequently, as in the work of Krasovskii [3,4] we shall take as an element of a trajectory of a system with retardation, not the vector function $x_i(x_0(\theta_0), t)$ at the instant of time t, but rather the vector-interval trajectory $x_i(x_0(\theta_0), t+\theta)$, where $-r < \theta \leqslant 0$. In view of this, the solution may naturally be considered as a trajectory in the function space C. We shall use the notation $x(t+\theta) = \{x_{it}(\theta)\}$, where $x_{it}(\theta)$, for fixed i and t, represents a point in the function space C, i.e. a function defined on the interval $-r < \theta \leqslant 0$.

In the function space the system of equations (1.1) is just a system of "ordinary" differential equations with an operator right-hand side [3]:

$$\frac{dx_t(\vartheta)}{dt} = R(x_t(\vartheta), t) \tag{1.2}$$

where

$$x_t(\vartheta) = \{x_{1t}(\vartheta), \ldots, x_{nt}(\vartheta)\} = \{x_1(t+\vartheta), \ldots, x_n(t+\vartheta)\}$$

$$R(x_t(\vartheta), t) = \begin{cases} \dfrac{dx_t(\vartheta)}{d\vartheta} & (-\tau \leqslant \vartheta \leqslant 0) \\ X(x_{1t}(\vartheta), \ldots, x_{nt}(\vartheta), t) & (\vartheta = 0) \end{cases}$$

The operator R is defined for $t \geqslant \tau + t_0$ when the initial function $x_0(\theta)$ is piecewise continuous, and it is defined for $t \geqslant t_0$ when the initial function is differentiable (at the instant $t = t_0$ by a derivative one must understand the right-hand derivative).

In order to solve the problem of the instability of the motion $x_i(\theta) \equiv 0$ one must: (1) establish instability criteria for unperturbed motions when subjected to arbitrary, small in norm, differentiable initial disturbances; (2) determine the domain of instability in the space of differentiable functions, such that the functions belonging to this domain produce solutions of increasing norms, rendering the solution $x \equiv 0$ unstable; (3) determine whether a given possible perturbation of the real system belongs or not to the domain of instability.

In the present paper it will be assumed that all perturbations which are differentiable with respect to t and are small in norm are possible perturbations. Liapunov's method permits the determination of the domain of instability in function space. Of course, the question as to whether a certain possible perturbation of the real system belongs to the domain of instability cannot be answered in general, since there are no general criteria for the choice of such perturbations.

Consider the functional $v(x_1(\theta), \ldots, x_n(\theta), t)$, defined for all piecewise continuous functions

$$\{x_1(\vartheta), \ldots, x_n(\vartheta)\} \qquad (-\tau \leqslant \vartheta \leqslant 0)$$

in the domain

$$\|x(\vartheta)\| < H, \qquad t \geqslant t_0 \tag{1.3}$$

As in Liapunov's theory, we shall suppose that the functional $v(x(\theta), t)$ is continuous with respect to $x(\theta)$ and t. For each continuous function $x_t(\theta)$ in the domain (1.3), the functional $v(x(\theta), t)$ is a continuous function of the time $v(t) = v(x_t(\theta), t)$. The following definitions, relative to functionals v, will be employed:

(1) A functional $v(x(\theta), t)$ is said to be positive definite, provided that there is a continuous function $w(r)$ satisfying the following conditions [2]:

$$v(x_1(\vartheta), \ldots, x_n(\vartheta), t) \geqslant w(\|x(\vartheta)\|) \tag{1.4}$$
$$w(r)\,r > 0, \qquad 0 < r \leqslant H, \qquad w(0) = 0$$

(2) A functional $v(x(\theta), t)$ admits an infinitely small upper bound in the domain (1.3), provided that there is a continuous function $w_1(r)$, satisfying the conditions

$$|v(x_1(\vartheta), \ldots, x_n(\vartheta), t)| \leqslant w_1(\|x(\vartheta)\|) \tag{1.5}$$
$$w_1(r)\,r > 0, \qquad 0 < r \leqslant H, \qquad w_1(0) = 0$$

(3) By the domain $v > 0$ we shall understand the set of piecewise continuous functions $x(\theta)$ which satisfy the conditions

$$\|x(\vartheta)\| \leqslant H, \qquad v(x(\vartheta), t) > 0, \qquad t \geqslant t_0 \tag{1.6}$$

In particular, if v is an operator of constant sign, according to the definition (1), then the domain $v > 0$ coincides with the domain (1.3).

(4) A functional v_1 admits an infinitely small upper bound in the domain $v > 0$ provided that it is bounded in the domain $v > 0$ and is such that for each positive ϵ, chosen arbitrarily, there is a number λ, different from zero, having the property that if

$$t > t_0, \qquad \|x(\vartheta)\| < \lambda, \qquad v \geqslant 0 \tag{1.7}$$

then

$$|v_1(x(\vartheta), t)| < \varepsilon$$

(5) A functional v_1 will be said to be of constant sign in the domain $v > 0$ provided that, for each positive ϵ, no matter how small, there is a number η, different from zero, such that whenever $x(\theta)$ satisfies the inequality $v > \epsilon$, one has also the following inequality:

$$|v_1(x(\vartheta), t)| \geqslant \eta$$

An example of a functional of constant sign is

$$v^0 = \lambda v + w \qquad (\lambda > 0)$$

where $w(x(\theta))$ is a positive functional in the domain $v > 0$, or else it is identically zero.

(6) A functional $v_1(x(\theta), t)$ is called the lower functional derivative of the functional $v(x(\theta), t)$ with respect to the system (1.2) provided that for any solution $x_t(\theta)$ of the system (1.2) which obeys (1.6) one has the equality

$$\lim\inf\left(\frac{\Delta v}{\Delta t}\right)_{(1.1)} = v_1(x_t(\vartheta), t) \quad \text{for } \Delta t \to +0 \tag{1.8}$$

where the subscript (1.1) indicates the number of the system along whose trajectories one calculates the limit $\Delta v/\Delta t$.

Theorem 1. If the differential equation (1.2) of the perturbed motion is such that there is a functional $v(x(\theta), t)$ which is bounded and has an infinitely small upper bound in the domain (1.6); and which is defined for $t \geqslant t_0$ and any arbitrarily small, in norm, differentiable function $x(\theta)$; and also that the lower functional derivative $v_1(x(\theta), t)$ with respect to the system (1.2) is positive definite on the domain $v > 0$ (see (1.6)), then the unperturbed motion $x_t(\theta) \equiv 0$ is unstable.

Proof: Let $v(x(\theta), t)$ be a functional satisfying the hypotheses of the theorem. Then in the domain (1.6) one has the inequality

$$v(x(\vartheta), t) < L \tag{1.9}$$

for some positive constant L.

It is to be proved that for no given positive number $H_1 < H$ does there exist a number λ, sufficiently small, such that the inequality $\|x_0(\theta)\| < \lambda$ implies the inequality $\|x_t(\theta)\| < H_1$ for $t \geqslant t_0$. In order to do this it is enough to show that, for any given λ there is at least one function $x_0(\theta)$ for which, at a certain instant $t = t_1$, the equality $\|x_{t_1}(\theta)\| = H_1$ holds.

Let us suppose the contrary. That is, let us suppose that there is a sufficiently small number λ having the property that if

$$\|x_0(\vartheta)\| < \lambda \qquad (\lambda < H_1)$$

then

$$\|x_t(\vartheta)\| < H_1 \qquad (t \geqslant t_0) \tag{1.10}$$

Let us choose an initial differentiable perturbation $x_0(\theta)$ such that

$$v(x_0(\vartheta), t_0) = v_0 > 0, \qquad \|x_0(\vartheta)\| = \lambda_1 < \lambda < H_1 \tag{1.11}$$

As long as the point $x_t(\theta)$ never leaves the domain (1.6), it follows that

$$v(x_t(\vartheta), t) \leqslant v_0 + \int_{t_0}^{t} v_1(x_t(\vartheta), t)\, dt < l_1(t - t_0) + v_0 \tag{1.12}$$

Since the functional v has an infinitely small upper bound in the domain $v > 0$, the conditions $v(x_t(\theta), t) > v(x_0(\theta), t_0) = v_0 > 0$ imply that, as long as the point $x_t(\theta)$ does not leave the domain $v > 0$, one will have $\| x_t(\theta) \| > \mu$, where μ is a positive number. But, on the domain $H_1 \geqslant \| x_t(\theta) \| \geqslant \mu$, $v \geqslant v_0$, the lower functional derivative v_1, which is positive definite on the domain $v > 0$, will satisfy the inequality $v_1(x_t(\theta), t) \geqslant l_1$, where l_1 is a positive number. Thus the inequality (1.12) holds, and this contradicts (1.9). From this it follows that the motion $x_t(\theta)$ with an initial function satisfying the condition (1.11) will, at some instant, leave the domain

$$\| x_t(\vartheta) \| < H_1, \qquad v(x_t(\vartheta), t) > 0, \qquad t \geqslant t_0$$

In view of the continuity with respect to t of $x_t(\theta)$ there must be at t_1 such that $\| x_{t_1}(\theta) \| = H_1$, and hence the motion $x \equiv 0$ is unstable.

(*Note*. If the initial perturbation is piecewise continuous then the following two possibilities arise for the motion $x_t(\theta)$:

1. As the time r increases the motion $x_r(\theta)$ belongs to the domain $v > 0$. Then $x_r(\theta)$ will also be differentiable with respect to r and it may be asserted that $x_r(\theta)$ belongs to the domain of instability. In this case the corresponding initial piecewise continuous function belongs to the domain of instability.

2. As the time r increases, the motion $x_r(\theta)$ does not belong to the domain $v > 0$. In this case we shall say that $x_r(\theta)$ and $x_0(\theta)$ do not belong to the domain of instability (even if $v_0(x_0(\theta), t_0) > 0$).

It is easy to obtain, from the theorem just proved, the following two theorems on the stability of the motion of systems with retardation, which are generalizations of the theorems of Liapunov on the stability of motion.)

Theorem 2 (first theorem of Liapunov concerning stability). If the differential equation of the perturbed motion (1.2) is such that there exists a functional $v(x(\theta), t)$ which possesses a positive definite lower functional derivative $v_1(x(\theta), t)$ with respect to the system (1.2), the functional v has an infinitely small upper bound; and if for any $t \geqslant t_0$, by a suitable choice of a differentiable, sufficiently small in norm, initial function $x_0(\theta)$, the functional $v(x(\theta), t)$ has the same sign as v_1, then the motion $x \equiv 0$ is unstable.

Theorem 3 (second theorem of Liapunov concerning stability). If the differential equation of the perturbed motion (1.2) is such that there exists a bounded functional $v(x(\theta), t)$ which possesses a lower functional derivative $v_1(x(\theta), t)$ with respect to (1.2) of the form:

$$v_1(x(\vartheta), t) = \lambda v(x(\vartheta), t) + w(x(\vartheta), t)$$

where λ is a positive constant, and w is either identically zero or is a positive definite functional; and, if the second alternative holds, the functional $v(x(\theta), t)$ is such that at $t > t_0$, by a suitable choice of the function $x_0(\theta)$, with sufficiently small norm $\| x_0(\theta)\|$, it can be made positive, then the unstable motion $x \equiv 0$ is unstable.

2. On the instability of motion, in the first approximation, for systems with after effects. Consider the equation of the perturbed motion

$$\frac{dx(t)}{dt} = \int_{-\tau}^{0} x(t+\vartheta)\, d\eta(\vartheta) + X(x(t+\vartheta), t) \qquad (2.1)$$

where $X(x(t+\theta), t)$ is a functional, which is defined for piecewise continuous functions $x(\theta)$ which are defined on the interval $-r < \theta < 0$.

The integral on the right hand side of equation (2.1) is a Stieltjes integral. If $d\eta(\theta) = 0$ for $\theta \neq 0$ and $\theta \neq -r$, and $d\eta(0) = a_1$, and $d\eta(-r) = a_2$, one obtains the following first approximation equation with retardation:

$$\frac{dx(t)}{dt} = a_1 x(t) + a_2 x(t-\tau) \qquad (2.2)$$

We shall suppose that the functional $X(x(t+\theta), t)$ satisfies a Lipschitz condition with respect to $x(\theta)$:

$$\| X(x_1(\vartheta), t) - X(x_2(\vartheta), t)\| < q \| x_1(\vartheta) - x_2(\vartheta)\| \qquad (2.3$$

where q is a positive number, whenever $\| x_1(\theta)\|$ and $\| x_2(\theta)\|$ satisfy

$$q < \left[\| x_1(\vartheta)\| + \| x_2(\vartheta)\|\right]^{\alpha} \qquad (\alpha > 0) \qquad (2.4)$$

The functional X is supposed to be continuous with respect to t for $t \geqslant t_0 > 0$. Consider the characteristic equation

$$\Delta(\lambda) \equiv -\lambda + \int_{-\tau}^{0} e^{\lambda\vartheta} d\eta(\vartheta) = 0 \qquad (2.5)$$

Theorem 2.1. If the equation (2.5) has at least one root possessing a positive real part then the undisturbed motion $x = 0$ of the system (2.1) is unstable, no matter what the functional X happens to be.

(*Note 1.* $\Delta(\lambda)$ is an entire function, and it may be expanded in a power series with majorant $|\lambda| + A \exp r|\lambda|$ (here A denotes the total variation of the function $\eta(\theta)$ on the interval $(-r, 0)$).

It is well known that the zeros of an entire function are only finite in number in any bounded domain, and that the only limit of these zeros is infinity [5,4].

The function $\Delta(\lambda)$ behaves in a similar manner as the polynomial to which it reduces in the particular case of the system with retardation, (2.2). In fact, the function $\Delta(\lambda)$ has a finite number of zeros on the right λ half-plane and on the imaginary axis. Indeed, in order to see this one has only to verify that there is a circle of radius R, outside of which $|\Delta(\lambda)|$ is bounded below by a positive number on the right half-plane, including the imaginary axis.

We have

$$|\Delta(\lambda)| = |\lambda|\cdot\left| -1 + \int_{-\tau} \frac{e^{\lambda\vartheta}}{\lambda}\, d\eta(\vartheta)\right| > \tfrac{1}{2} R$$

where R is sufficiently large, provided that, for λ in the right half-plane, the following inequality holds:

$$\left|\int_{-\tau}^{0} \frac{e^{\lambda\vartheta}}{\lambda}\, d\eta(\vartheta)\right| < \tfrac{1}{2}$$

The preceding inequality does indeed hold, because the function $\exp \lambda\theta$ is bounded in absolute value on the right half-plane, including the imaginary axis, and hence the absolute value $|\lambda|$ can be chosen sufficiently large, for R large enough.

Note 2. For the sake of simplicity it will be supposed in the proof that the roots with positive real parts are all positive.)

Proof: The equations (2.1) of the perturbed motion are equivalent to the ordinary differential equation with an operator right-hand side:

$$\frac{dx_t(\vartheta)}{dt} = Ax_t(\vartheta) + R(x_t(\vartheta), t) \tag{2.6}$$

where

$$Ax_t(\vartheta) = \begin{cases} \dfrac{dx_t(\nu)}{d\vartheta} & (-\tau \leqslant \vartheta < 0), \\ \displaystyle\int_{-\tau}^{0} x_t(\vartheta)\, d\eta(\vartheta) & (\vartheta = 0), \end{cases} \qquad R(x_t(\vartheta), t) = \begin{cases} 0 & (-\tau \leqslant \vartheta < 0) \\ X(x_t(\vartheta), t) & (\vartheta = 0) \end{cases}$$

Suppose that the roots with positive real parts are $\lambda_1, \ldots, \lambda_k, \ldots, \lambda_l$. Consider l linear functionals, corresponding to these roots:

$$f_k[x_t(\vartheta)] \equiv x_t(0) - \int_{-\tau}^{0} \left[\int_{0}^{\vartheta} e^{(\vartheta-\xi)\lambda k}\, x_t(\xi)\, d\xi\right] d\eta(\vartheta) \qquad (k = 1, \ldots, l) \tag{2.7}$$

Let us suppose that the initial functions of the solution are differentiable for $-r \leqslant t \leqslant 0$. Then the operator equation (2.6) holds for $t > 0$ (for $t = 0$, by dx/dt is meant the right hand derivative with respect to t). Hence the functional (2.7) fulfils

$$f_k[Ax(\vartheta)] \equiv \lambda_k f_k[x(\vartheta)] \qquad (k = 1, \ldots, l) \tag{2.8}$$

$$f_k[x_t(\vartheta)] = f_k[x_0(\vartheta)] \exp \lambda_k t \qquad (t \geqslant 0) \tag{2.9}$$

for differentiable $x_0(\theta)$ and solutions $x_t(\theta)$ of the first approximation equations

$$\frac{dx_t(\vartheta)}{dt} = Ax_t(\vartheta) \tag{2.10}$$

Lemma: If the differentiable function $x_0(\theta)$ satisfies the l conditions

$$f_k[x_0(\vartheta)] = 0 \qquad (k = 1, \ldots, l) \tag{2.11}$$

then the corresponding solution $x_t(\theta)$ of (2.10) satisfies the l conditions

$$f_k[x_t(\vartheta)] = 0 \tag{2.12}$$

for $t > 0$; and, as t increases, decreases in norm, exponentially, with exponents

$$\theta\alpha = \min |\operatorname{Re} \lambda_j| \qquad (j = l + 1, \ldots)$$

where the remaining roots $\lambda_{l+1}, \lambda_{l+2}, \ldots$ have negative real parts, and θ is positive number less than unity. (The case of purely imaginary roots can be reduced to the previous one by the transformation $y \exp(\beta t) = x$.)

The proof of the lemma may be carried out analogously to the proof of Theorem 29.1 of Reference [3], because the expression for the semi-group-operator on the class of functions satisfying conditions (2.11) is given by formula (29.29) of Reference [3]. Consider the subspace L, of the space of functions $x_t(\theta)$, consisting of all functions which satisfy the conditions

$$f_k[x_t(\vartheta)] = 0 \qquad (k = 1, \ldots, l)$$

Then every element $x_t(\theta)$ of the original function space may be represented in the form

$$x_t(\vartheta) = z_t(\vartheta) + y \qquad (z_t(\vartheta) \in L, \ y \in l)$$

which decomposes the space into two subspaces:

$$y_k = f_k[x_t(\vartheta)] \quad (k = 1, \ldots, l), \qquad x_t(\vartheta) = z_t(\vartheta) + \frac{\exp \lambda_1 \vartheta}{\Delta'(\lambda_1)} y_1 + \ldots + \frac{\exp \lambda_l \vartheta}{\Delta'(\lambda_l)} y_l \tag{2.13}$$

It is clear that $f_k[z_t(\theta)] = 0$, since

$$[\Delta'(\lambda_j)]^{-1} f_k[\exp \lambda_j \vartheta] = \begin{cases} 1 & (j = k) \\ 0 & (j \neq k) \end{cases}$$

Thus the system (2.10) leads to the system

$$\frac{dy_k}{dt} = \lambda_k y_k \qquad (k = 1, \ldots, l) \tag{2.14}$$

$$\frac{dz_t(\vartheta)}{dt} = Az_t(\vartheta), \qquad f_k[z_t(\vartheta)] = 0 \qquad (k = 1, \ldots, l) \tag{2.15}$$

Replacing x by y and z, in accordance with Equation (2.13), yields

$$\frac{dy_k}{dt} = \lambda_k y_k + f_k[R(x_t(\vartheta))] \qquad (k = 1, \ldots, l) \tag{2.16}$$

$$\frac{dz_t(\vartheta)}{dt} = Az_t(\vartheta) + R(x_t(\vartheta), t) - \Delta'(\lambda_1)^{-1}\exp(\lambda_1 \vartheta) f_1[R(x, t)] - \cdots$$
$$\cdots - \Delta'(\lambda_l)^{-1}\exp(\lambda_l\vartheta) f_l[R(x_t(\vartheta), t)] \tag{2.17}$$

$$f_k[z_t(\vartheta)] = 0 \qquad (k = 1, \ldots, l).$$

where

$$x_t(\vartheta) = z_t(\vartheta) - \frac{\exp\lambda_1\vartheta}{\Delta'(\lambda_1)} y_1 - \cdots - \frac{\exp\lambda_l\vartheta}{\Delta'(\lambda_l)} y_l$$

Obviously, the systems (2.16), (2.17) represent the system (2.6) in the subspaces $z_t(\theta)$ and $y(t)$.

Since the decomposition of $x_t(\theta)$ into $y(t)$ and $z_t(\theta)$ is unique, the fact that $x_t(\theta) = 0$ implies that $y_k = 0$ and that $z_t(\theta) = 0$. Thus the solutions of the systems (2.16) and (2.17) exist and are unique, provided only that the solution of the system (2.6) corresponding to $x_0(\theta)$ exists and is unique.

Let us consider the system (2.14). For this system the function v_1 may be constructed as follows: $v_1 = \lambda_1 y_1 \bar{y}_1 + \ldots + \lambda_l y_l \bar{y}_l$.

The function v_l is positive definite in the finite dimensional y subspace. The derivative of v_1 along an arbitrary trajectory of the system (2.14) is positive definite on the subspace y:

$$\left(\frac{dv_1}{dt}\right)_{(2.14)} = \lambda_1[(\lambda_1 + \bar{\lambda}_1) y_1\bar{y}_1] + \cdots + \lambda_l[(\lambda_l + \bar{\lambda}_l) y_l\bar{y}_l] \tag{2.18}$$

Further, on the subspace L, for the linear system (2.15), one may construct (in view of the results of [3], pp. 191-192) a functional $v_2(z_t(\theta))$ satisfying the following conditions:

$$c_1\|z_t(\vartheta)\| < v_2(z_t(\vartheta), t) < c_2\|z_t(\vartheta)\| \tag{2.19}$$

$$\limsup\left(\frac{\Delta v_2}{\Delta t}\right)_{(2.15)} \leqslant -c_3\|z_t(\vartheta)\| \qquad \text{for } \Delta t \to +0 \tag{2.20}$$

$$|v(z_t^{\bullet}(\vartheta), t) - v(z_t'(\vartheta), t)| \leqslant c_4\|z_t^{\bullet}(\vartheta) - z_t'(\vartheta)\| \tag{2.21}$$

where c_1, c_2, c_3, c_4 are positive numbers.

We now form the functional $v^*(x_t(\theta),\ t)$ for the complete system as follows:

$$v^*(x_t(\vartheta),\ t) = 2\sqrt{v_1(y)} - v_2(z_t(\vartheta),\ t)] \tag{2.22}$$

where y and $z_t(\theta)$ are expressed in terms of $x_t(\theta)$ by formula (2.13).

It is readily verified that the functional v^* satisfies all the hypotheses of the first stability theorem of Liapunov for systems with after-effects (Theorem 2). Let us now compute the lower functional derivative of the functional v^* with respect to the system (2.6) (or what is the same, the systems (2.16) and (2.17)). We have

$$\begin{aligned}
-\lim_{\Delta t\to+0}\inf\left(\frac{\Delta v^*}{\Delta t}\right)_{(2.6)} &= -\left(\frac{1}{\sqrt{v_1}}\frac{dv_1}{dt}\right)_{(2.6)} - \lim_{\Delta t\to+0}\inf\left(\frac{-\Delta v_2}{\Delta t}\right)_{(2.6)} = \\
&= -\left(\frac{1}{\sqrt{v_1}}\frac{dv_1}{dt}\right)_{(2.6)} + \lim_{\Delta t\to+0}\sup\left(\frac{\Delta v_2}{\Delta t}\right)_{(2.6)} = \\
&= -\frac{1}{\sqrt{v_1}}\sum_{j=1}^{l}\lambda_j 2\,\mathrm{Re}\,\lambda_j\, y_j\bar{y}_j + \lim_{\Delta t\to+t_0}\sup\left(\frac{\Delta v_2}{\Delta t}\right)_{(2.15)} + \\
&+ \frac{1}{\sqrt{v_1}}\sum_{j=1}^{l}\lambda_j\,[y_j\bar{f}_j(X) + \bar{y}_j\cdot f(X)] + \lim\left[\left(\frac{\Delta v_2}{\Delta t}\right)_{(2.6)} - \left(\frac{\Delta v_2}{\Delta t}\right)_{(2.15)}\right] < \\
&< -c\|y\| - c\|z_t(\vartheta)\| + \frac{lq}{c_6}\max|\lambda_j|\ \|x_t(\vartheta)\| + c_4 q\|x_t(\vartheta)\| < \\
&< (-c + Dq)\|x_t(\vartheta)\| < -\tfrac{1}{2}c\,\|x_t(\vartheta)\|
\end{aligned}$$

where

$$c_6\|y\| < \sqrt{v_1} < c_5\|y\|$$

$$c = \min\left\{\frac{1}{c_5}\min\left|\sum_{j=1}^{n}\lambda_j\ 2\,\mathrm{Re}\,\lambda_j\ y\bar{y}\right|\ (\text{or}\ \|y\| = 1),\ c_3\right\}$$

$$D = \frac{l\max|\lambda_j|}{c_6} + c_4,\qquad \|y\| + \|z_t(\vartheta)\| \geqslant \|x_t(\vartheta)\|,\qquad c > 2qD\ \text{for}\ \|x_t(\vartheta)\| < H_1$$

Thus, on account of the inequality

$$\lim_{\Delta t\to+0}\inf\left(\frac{\Delta v^*}{\Delta t}\right)_{(2.6)} > \frac{c}{2}\|x_t(\vartheta)\| \tag{2.23}$$

it follows that the lower functional derivative of v^* with respect to the system (2.6) is positive definite for sufficiently small $\|x_t(\theta)\|$.

Consider now the function

$$x_0^*(\vartheta) = \sum_{j=1}^{l} \frac{\exp \lambda_j \vartheta}{\Delta'(\lambda_j)} \eta \qquad (-\tau \leqslant \vartheta \leqslant 0)$$

where η is a positive number, so chosen that $\| x_0^*(\theta) \|$ is as small as is desired.

To the function $x_0^*(\theta)$ correspond the functions $z_0^*(\theta) \equiv 0$, $y_j = \eta \exp \lambda_j \theta$ $(j = 1, \ldots, l)$ in the spaces L and l, and from (2.18) and (2.22) it follows that

$$v^*(x_0^*(\vartheta)) > 0 \tag{2.24}$$

Now, the norms of y_k and $z_t(\theta)$, from the definitions (2.13) and (2.7), are found to satisfy

$$\| y_k \| < A \| x_t(\vartheta) \|, \qquad \| z_t(\vartheta) \| < B \| x_t(\vartheta) \| \tag{2.25}$$

where A and B are positive numbers. Let us now estimate the norm of v^*. Taking into account (2.25), (2.19), (2.18), and (2.22), we obtain

$$\| v^* \| < N \| x_t(\vartheta) \|, \qquad N = l \max \{ | \lambda_j | \} \sqrt{2} + c_2 B \tag{2.26}$$

where N is a certain positive number. From (2.26) it is seen that the functional v^* has an infinitely small upper bound on any domain $\| x_t(\theta) \| \leqslant H$, where H is a positive number.

Thus, the functional v^* possesses a lower functional derivative with respect to the system (2.6), which is positive definite, in view of (2.23). Further, v^* possesses an infinitely small upper limit, by (2.26), and there exists a differentiable initial function $x_0(\theta)$, arbitrarily small in norm, for which v^* assumes positive values (see (2.24)). Therefore v^* satisfies all the hypotheses of the first Liapunov theorem on the stability of motion (Theorem 2). Consequently, the motion $x = 0$ of the system (2.1) is unstable for arbitrary X (fulfilling the required conditions).

(*Note 3.* It is not difficult to prove the theorem in the case of non-simple roots λ_j with positive real parts.

Note 4. The theorem concerning the stability of the motion $x = 0$ is valid also for systems of n equations of the form

$$\frac{dx_i(\vartheta)}{dt} = \sum_{j=1}^{n} \int_{-\tau}^{0} x_j(t+\vartheta)\, d\eta_{ij}(\vartheta) + X_i(x_1(t+\vartheta), \ldots, x_n(t+\vartheta), t)$$

where $d\eta_{ij}(\theta)$ and X_i play roles similar to those of $d\eta(\theta)$ and X in the system (2.1). The proof of this assertion does not differ essentially from the proof of Theorem (2.1).

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